Kuwait University | **Math 102** | Date : 15th January 2007

Dept.Of Math & Comp.Sci. | **Final Exam** | Duration : 2 hours

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1. **(2+2 pts.).** Let $f(x)=\cos^{-1}\left(2e^{x}-3\right)$

   a. Determine the domain of f and show that f has an inverse.

   b. Find $f^{-1}(x)$ and state the domain of f^{-1}.

2. **(3 pts.).** Use logarithmic differentiation to find y' if $y=\dfrac{(\tanh x)^{\sinh x}}{\sqrt{3^{x}-x}}$, $x>0$.

3. **(3 pts.).** Find $\lim\limits_{x\to 0^{+}}\left(e^{x}-1\right)^{\frac{1}{\ln x}}$

4. **(3 pts. each).** Evaluate the following integrals:

   a. $\displaystyle\int\frac{\ln x}{\sqrt[3]{x}}\,dx$ b. $\displaystyle\int\frac{x^{2}+2x+5}{x^{4}-1}\,dx$ c. $\displaystyle\int\frac{3x+2}{\sqrt{4x-x^{2}}}\,dx$.

5. **(4 pts.).** Determine whether the improper integral $\displaystyle\int_{0}^{\infty}\frac{dx}{\left(x^{2}+1\right)^{2}}$ converges or diverges, and find its value if it converges.

6. **(4 pts.).** Let the curve $C: y=\frac{1}{8}x^{2}-\ln x,\ x\in[1,2]$. Find the area of the surface of revolution obtained by rotating C about y- axis

7. **(3 pts. each).** Let the curve $\Gamma: x=e^{t}\cos t,\ y=e^{t}\sin t,\ t\in[0,2\pi]$.

   a. Find the equation of the tangent line at the point corresponding to $t=\frac{\pi}{2}$.

   b. Find the points at which the tangent lines to Γ are vertical.

   c. Find the length of Γ .

8. **(4 pts.).** Find the area of the region that is inside the graphs of both equations :

   $r=2+2\cos\theta$ and $r=3$.

SOLUTIONS

1. a. $-1 \le 2e^x - 3 \le 1 \iff 1 \le e^x \le 2 \iff x \in [0, \ln 2]$. $f'(x) = -\frac{2e^x}{\sqrt{1-(2e^x-3)^2}} < 0$, so f is decreasing $\Rightarrow f$ is one-to-one $\Rightarrow \exists f^{-1}$.

b. $y = \cos^{-1}(2e^x - 3) \Rightarrow \cos y = 2e^x - 3 \Rightarrow e^x = \frac{1}{2}(3+\cos y) \Rightarrow x = \ln\frac{3+\cos y}{2}$.
So $f^{-1}(x) = \ln\frac{3+\cos x}{2}$. $\lim\limits_{x\to 0^+}\cos^{-1}(2e^x-3) = \pi$, $\lim\limits_{x\to(\ln 2)^-}\cos^{-1}(2e^x-3) = 0 \Rightarrow D_{f^{-1}} = [0, \pi]$.

2. $\ln y = (\sinh x)\ln(\tanh x) - \frac{1}{2}\ln(3^x - x) \Rightarrow y' = y\left\{(\cosh x)\ln(\tanh x) + \operatorname{sech} x - \frac{1}{2}\frac{3^x\ln 3 - 1}{3^x - x}\right\}$.

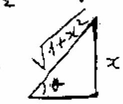
3. $y = (e^x - 1)^{\frac{1}{\ln x}}$; $\lim\limits_{x\to 0^+}\ln y = \lim\limits_{x\to 0^+}\frac{\ln(e^x-1)}{\ln x} \underset{L'H}{=} \lim\limits_{x\to 0^+}\frac{\frac{e^x}{e^x-1}}{\frac{1}{x}} = \lim\limits_{x\to 0^+}\frac{xe^x}{e^x-1}$
$\underset{L'H}{=} \lim\limits_{x\to 0^+}(1+x) = 1$. So, $\lim\limits_{x\to 0^+} y = e$.

4. a. Integrate by parts: $u = \ln x$, $dv = x^{-1/3}dx$, $du = \frac{dx}{x}$, $v = \frac{3}{2}x^{2/3}$.
$I = \frac{3}{2}x^{2/3}\ln x - \frac{3}{2}\int x^{-1/3}dx = \frac{3}{2}x^{2/3}\left(\ln x - \frac{3}{2}\right) + C$.

b. $\frac{x^2+2x+5}{x^4-1} = \frac{2}{x-1} - \frac{1}{x+1} - \frac{x+2}{x^2+1} \Rightarrow I = \ln\frac{(x-1)^2}{|x+1|} - \frac{1}{2}\ln(x^2+1) - 2\tan^{-1}x + C$

c. $I = \int\frac{3x+2}{\sqrt{4-(x-2)^2}}dx = \int\frac{3t+8}{\sqrt{4-t^2}}dt = 3\int\frac{t\,dt}{\sqrt{4-t^2}} + 8\int\frac{dt}{\sqrt{4-t^2}}$
$(x-2=t,\ dx=dt)$
$= -3\sqrt{4-t^2} + 8\sin^{-1}\left(\frac{t}{2}\right) + C = -3\sqrt{4x-x^2} + 8\sin^{-1}\left(\frac{x-2}{2}\right) + C$.

5. $\int\frac{dx}{(x^2+1)^2} = \int\cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta = \frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2} + C$.
$(x = \tan\theta,\ dx = \sec^2\theta\,d\theta)$
$\Rightarrow \lim\limits_{t\to\infty}\int_0^t\frac{1}{(x^2+1)^2}dx = \lim\limits_{t\to\infty}\left(\frac{1}{2}\tan^{-1}t + \frac{1}{2}\frac{t}{1+t^2}\right) = \frac{\pi}{4}$. So, int. conv.



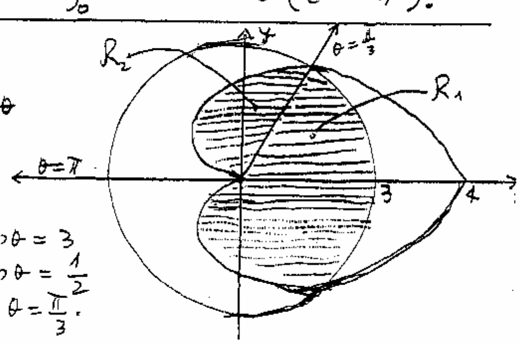
6. $y' = \frac{x}{4} - \frac{1}{x} \Rightarrow 1 + (y')^2 = 1 + \frac{x^2}{16} - \frac{1}{2} + \frac{1}{x^2} = \left(\frac{x}{4} + \frac{1}{x}\right)^2$. So: $S = \int_1^2 2\pi x\sqrt{1+(y')^2}\,dx$
$= 2\pi\int_1^2 x\left(\frac{x}{4} + \frac{1}{x}\right)dx = \frac{19\pi}{6}$.

7. a. $m = \frac{\sin t + \cos t}{\cos t - \sin t}$; for $t = \frac{\pi}{2} \Rightarrow m = -1$, $P_0(0, e^{\pi/2})$, so the tangent line has the equation $x + y - e^{\pi/2} = 0$.

b. Put $\cos t - \sin t = 0 \Rightarrow \tan t = 1 \Rightarrow t_1 = \frac{\pi}{4}, t_2 = \frac{5\pi}{4}$. Thus the points are:
$P_1\left(\frac{\sqrt{2}}{2}e^{\pi/4}, \frac{\sqrt{2}}{2}e^{\pi/4}\right)$ and $P_2\left(-\frac{\sqrt{2}}{2}e^{5\pi/4}, -\frac{\sqrt{2}}{2}e^{5\pi/4}\right)$.

c. $\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t}(\cos t - \sin t)^2 + e^{2t}(\cos t + \sin t)^2 = 2e^{2t}$.
So: $L = \int_0^{2\pi}\sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \sqrt{2}\int_0^{2\pi}e^t\,dt = \sqrt{2}\left(e^{2\pi} - 1\right)$.

8. Area $\mathcal{R} = 2\left(\text{Area } R_1 + \text{Area } R_2\right)$
$= \int_0^{\pi/3}9\,d\theta + 4\int_{\pi/3}^{\pi}(1+\cos\theta)^2\,d\theta$
$= 7\pi - \frac{9\sqrt{3}}{2}$.



$2 + 2\cos\theta = 3$
$\cos\theta = \frac{1}{2}$
$\theta = \frac{\pi}{3}$.